An Introduction to Extremal Graph Theory

Anna Lladó

Dept. Matemàtica Aplicada 4
Universitat Politècnica de Catalunya
Spain

CIMPA-UNESCO-Indonesia School
Extremal problems and Hamiltonicity in Graphs

Program

1. The problem of forbidden subgraphs.
   The famous Turán theorem.
2. The fundamental Erdős-Stone theorem.
   The Regularity Lemma
3. Extremal problems for paths and cycles
4. Extremal problems on the order
5. Extremal problems on connectivity
6. Some extremal exercises.

Introduction

- Extremal Graph Theory started with the following question:

  What is the **minimum size**
  of a graph with a given order to ensure
  that it **contains a triangle** as a subgraph?

  (This problem was solved by Mantel in 1907).

Introduction

- Extremal Graph Theory started with the following question:

  What is the **minimum size**
  of a graph with a given order to ensure
  that it **contains a triangle** as a subgraph?

  (This problem was solved by Mantel in 1907).

- One equivalent way to consider this question is:

  What is the **maximum size**
  of a graph with a given order such that
  it does NOT **contain a triangle** as a subgraph?

- In 1941, the famous hungarian mathematician Paul Turán gave a much more general answer to this question.

What is the maximum size of a graph with a given order n,
such that
it does not no contain the complete graph K_r, $2 \leq r \leq n$,
as a subgraph?

Introduction

The **Theorem of Turán** was the origin of what it is known as

The problem of forbidden subgraphs:

What is the maximum size of a graph with a given order such that it does not contain a prescribed subgraph F?

In this context, the maximum size of G is denoted by

$$ex(n; F)$$

and the set of graphs with this density, that is, the extremal graphs, by

$$EX(n; F).$$

Introduction

- When the subgraph $F \neq K_r$
  the problem becomes much more difficult.

- The important Theorem of **Erdős-Stone**, 1946,
  which deals with multipartite complete graphs,
  stimulated the progress on the problem.

- In 1967 **Erdős and Simonovits** went further to the problem
  using chromatic partitions and they verify that
  the extremal graphs with a large order
  are very close to the "extended" Turán graphs.

Introduction

- The problem of the existence of **long paths and cicles** can be very difficult.

  Some of them are NP-complete problems.

  Here we are going to present some partial results.

Introduction

There are a lot of problems/questions that fit
in what it is called Extremal Graph Theory.

When we are looking for graphs with **some property**,
we can **fix some parameters** on the graphs and
attack the problem
looking for the maximum/minimum value
of the prescribed parameter (property)
depending only on the value of the fixed parameters.

- A **big problem** in extremal graph theory is to determine the existence of graphs with maximum/minimum **order** with respect two fixed invariants on them:

  The Moore graphs

  .

  (the extremal graphs)

- A very important property for a graph is its **connectivity**.

  When we try to characterize the graphs with
  connectivities larger than 2
  the problem becomes **really hard**.

  We will give some extremal results in this direction.

- We will finish with some extremal exercises.

Mantel's Theorem

We are going to deal with simple graphs, $G = (V, E)$

$$|V(G)| = n, \qquad |E(G)| = m$$

Theorem

[Mantel, 1907] *Any connected graph of order $n \geq 3$ and size*

$$m > \lfloor \frac{n^2}{4} \rfloor$$

has a triangle.

- **Remark:** G must be connected:

  $G = K_3 \cup \{x\}$ has 1 triangle but $3 < \frac{4^2}{4}$

  $G = K_3 \cup K_3$ has 2 triangles but $6 < \frac{6^2}{4}$.

Proof of Mantel's Theorem (I)

Proof:
We use induction on **n**.

- The unique graph with $n = 3$ and $m > \lfloor \frac{9}{4} \rfloor$ is

$$G = K_3$$

- The unique graphs with $n = 4$ and $m > \lfloor \frac{16}{4} \rfloor$ are

$$G = K_4 - e, \qquad G = K_4$$

- Let G be a graph of order $n \geq 5$ and size $m > \lfloor \frac{n^2}{4} \rfloor$.

  Suppose as IH:
  any graph G' of order $3 \leq n' < n$ and size $m' > \lfloor (n')^2/4 \rfloor$
  contains a triangle.

Proof of Mantel's Theorem (II)

We want to deduce that G also has a triangle.

Let $xy \in E(G)$.

- If there is $z \in \Gamma(x) \cap \Gamma(y)$, then x, y and z form our triangle.
- If $\Gamma(x) \cap \Gamma(y) = \emptyset$, we consider the graph

$$G' = G \setminus \{x, y\}.$$

  - If G' has size $m' > \lfloor \frac{(n-2)^2}{4} \rfloor$ then,
    by the IH:
    $K_3 \hookrightarrow G'$ and therefore $K_3 \hookrightarrow G$.

Proof of Mantel's Theorem (and III)

- Suppose, G' has size $m' \leq \lfloor \frac{(n-2)^2}{4} \rfloor$.

  Since $\Gamma(x) \cap \Gamma(y) = \emptyset$,
  any $z \in V(G')$ is adjacent to **at most** one of the 2 vertices x o y.

  So, when we remove x and y
  we remove **at most** $(n-2) + 1(xy)$ edges.

  Therefore, since $m > \lfloor \frac{n^2}{4} \rfloor$,
  we have the following contradiction:

$$m' > \lfloor \frac{n^2}{4} \rfloor - (n-1) = \lfloor \frac{(n-2)^2}{4} \rfloor.$$

□

Some Remarks on Mantel's Theorem

Remark: This condition is not necessary:

2 triangles sharing a vertex has $n = 5$, $m = 6$ but

$$6 < \frac{5^2}{4}.$$

Remark: The bound in Mantel's theorem is tight:

The Theorem of Turán will confirm that **the only** graph with order n and size $\lfloor n^2/4 \rfloor$ which that does not contain a triangle is

$$K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$$

On the number of triangles

We can obtain the absolute **lower bound** for the number of triangles of any graph in terms of its order and size.

Theorem

The number $t_3(G)$ of triangles of a graph $G(n,m)$ verifies

$$t_3(G) \geq \frac{m}{3n}(4m - n^2).$$

Remark

This bound is tight.

Exercise

Find the extremal graphs for this bound.

On the number of triangles: Proof of lower bound (I)

Proof:

Let $d_1, \leq d_2 \leq \ldots \leq d_n$ be the degree sequence of G.

Denote by

$t_i(G)$: the number of triples of $V(G)$ with i edges, $0 \leq i \leq 3$.

$t_i(\bar{G})$: the number of triples of $V(G)$ with $0 \leq i \leq 3$ edges in $\bar{G}$.

First we prove that

$$t_3(G) + t_3(\bar{G}) = \binom{n}{3} - (n-2)m + \sum_{i=1}^{n} \binom{d_i}{2}.$$

On the number of triangles: Proof of a lower bound (II)

First we prove that

$$t_3(G) + t_3(\bar{G}) = \binom{n}{3} - (n-2)m + \sum_{i=1}^{n} \binom{d_i}{2}.$$

- The number of triples in $V(G)$ is

$$\binom{n}{3} = t_0(G) + t_1(G) + t_2(G) + t_3(G).$$

- For each edge of $E(G)$,

  the number of triples that contains this edge is,

$$m(n-2) = t_1(G) + 2t_2(G) + 3t_3(G).$$

- and the number of pairs of adjacent edges is,

$$\sum_{i=1}^{n} \binom{d_i}{2} = t_2(G) + 3t_3(G). \qquad (1)$$

On the number of triangles: Proof of lower bound (II)

Then,

$$t_0(G) + t_3(G) = \binom{n}{3} - m(n-2) + \sum_{i=1}^{n} \binom{d_i}{2}. \qquad (2)$$

Then, $t_3(\bar{G}) = t_0(G)$.

Therefore,

$$t_3(G) + t_3(\bar{G}) = \binom{n}{3} - m(n-2) + \sum_{i=1}^{n} \binom{d_i}{2}. \qquad (3)$$

On the number of triangles: Proof of lower bound (II)

Similarly

as we did in the equation (3), in $\bar{G}$ we have

$$t_3(\bar{G}) + t_3(G) = \binom{n}{3} - \bar{m}(n-2) + \sum_{i=1}^{n} \binom{\bar{d}_i}{2}. \quad (4)$$

The number of adjacent edges in $\bar{G}$ is,

$$\sum_{i=1}^{n} \binom{\bar{d}_i}{2} = t_2(\bar{G}) + 3t_3(\bar{G}).$$

On the number of triangles: Proof of lower bound (III)

Then,

$$t_3(\bar{G}) \leq \frac{1}{3}\sum_{i=1}^{n}\binom{\bar{d}_i}{2}.$$

Substituting in (4) we have,

$$t_3(G) \geq \binom{n}{3} - \bar{m}(n-2) + \frac{2}{3}\sum_{i=1}^{n}\binom{\bar{d}_i}{2}. \tag{5}$$

On the number of triangles: Proof of lower bound (IV)

One **lower bound** for $\sum_{i=1}^{n} \binom{\bar{d}_i}{2}$ is

$$\begin{aligned}
\sum_{i=1}^{n} \binom{\bar{d}_i}{2} &= \frac{1}{2}\Big(\sum_{i=1}^{n} \bar{d}_i^{\,2} - \sum_{i=1}^{n} \bar{d}_i\Big) \\
&\geq \frac{1}{2}\left(\frac{1}{n}\Big(\sum_{i=1}^{n} \bar{d}_i\Big)^2 - \sum_{i=1}^{n} \bar{d}_i\right) \; (*) \\
&= \frac{1}{2}\Big(\frac{4\bar{m}^2}{n} - 2\bar{m}\Big) (**) \\
&= \frac{\bar{m}}{n}(2\bar{m} - n).
\end{aligned}$$

Remark

(*)Cauchy-Schwarz inequality.
(**) Handshaking Lemma.

On the number of triangles: Proof of lower bound (and V)

Substituting in (5),

$$t_3(G) \geq \binom{n}{3} - \bar{m}(n-2) + \frac{2}{3}\frac{\bar{m}}{n}(2\bar{m} - n).$$

Finally, as $\bar{m} = \binom{n}{2} - m$,

$$t_3(G) \geq \frac{m}{3n}(4m - n^2).$$

□

Extension to larger cliques: first approximation

Next result generalizes Mantel's theorem for any complete graph.

Theorem

Let n and r be two natural numbers, $2 \leq r \leq n$.

A graph G of order n contains K_r if it has size

$$m > \frac{n^2}{2}\left(\frac{r-2}{r-1}\right).$$

Proof first approximation (I)

Proof:

We use induction on r.

- Any graph with an edge contains K_2.
- For $r = 3$, from Mantel's theorem we have

$$K_3 \hookrightarrow G$$

- For $4 \leq r \leq n$,
  IH: any graph G' of order n and size $m' > \frac{n^2}{2}(\frac{r-2}{r-1})$

$$K_{r-1} \hookrightarrow G'$$

Proof first approximation (II)

Now we use induction on n.

- If $n = r$, then
$$\frac{n^2}{2}(\frac{r-2}{r-1}) < m \leq \binom{n}{2}$$
and
$$G = K_r.$$
- Let $n > r \geq 4$
IH:
any graph G' of order k, $4 \leq r \leq k < n$, and size $m' > \frac{k^2(r-2)}{2(r-1)}$
$$K_r \hookrightarrow G'.$$

Proof first approximation (III)

Let G be a graph of order $n > r$ and size $m > \frac{n^2}{2}(\frac{r-2}{r-1})$.
Since

$$m > \frac{n^2}{2}(\frac{r-2}{r-1}) > \frac{n^2}{2}(\frac{r-3}{r-2}),$$

by the IH on r,

$$K_{r-1} \hookrightarrow G.$$

Now we want to see that

$$K_r \hookrightarrow G.$$

Proof first approximation (IV)

Now we want to see that

$$K_r \hookrightarrow G.$$

Let $X \subset V(G)$ s.t. $G[X] \simeq K_{r-1}$

Consider the subgraph $H = G - X$.

- If there is $x \in V(H)$ s.t. $\Gamma(x) \subset X$, then

$$K_r \hookrightarrow G.$$

- Suppose that $\forall x \in V(H)$, $|\Gamma(v) \cap X| \leq r-2$.
  Then,

$$|E(G)| < \binom{r-1}{2} + \binom{n-r+1}{2} + (r-2)(n-r+1).$$

Proof first approximation (V)

Then,

$$|E(G)| < \binom{r-1}{2} + \binom{n-r+1}{2} + (r-2)(n-r+1).$$

- If $n - r + 1 < r$, we have $n \leq 2(r-1)$ and we have the following contradiction:

$$\begin{aligned} |E(G)| &< 2\binom{r-1}{2} + (r-2)(r-1) \\ &= 2(r-1)(r-2) \\ &= \frac{r-2}{2r-2}n^2. \end{aligned}$$

- Therefore, $n - r + 1 \geq r$.

Proof first approximation (and VI)

On the other hand,

$$\begin{aligned} |E(H)| &> \frac{r-2}{2r-2}n^2 - \binom{r-1}{2} - (r-2)(n-r+1) \\ &= \frac{r-2}{2r-2}n^2[n^2-(r-1)^2-2(r-1)(n-r+1)] \\ &= \frac{r-2}{2r-2}(n-(r-1))^2. \end{aligned}$$

From the IH on n we have

$$K_r \hookrightarrow H$$

therefore

$$K_r \hookrightarrow G.$$

□

Last remarks on first approximation

Exercise

Prove that the bound of the above Theorem is not tight. □

Corollary

Let n and r two natural numbers $2 \leq r \leq n$.
Any graph of order n which does not contains K_r has size

$$m \leq \frac{n^2}{2}\left(\frac{r-2}{r-1}\right).$$

□

The Theorem of Turán

Turán graphs

Given two natural numbers $2 \leq r \leq n$,
what is the maximum number of edges in a graph G of a given order n s.t.

$$K_{r+1} \not\to G?$$

This number of edges is denoted by

$$t_r(n)$$

The graphs which attain this density are called Turán graphs:

$$T_r(n).$$

The Theorem of Turán

- Clearly $T_1(n) = N_n$.
- Mantel's theorem asserts that

$$t_2(n) = \lfloor \frac{n^2}{4} \rfloor$$

We already have seen that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is an extremal graph.

In particular,
Turán's Theorem proves that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is **the only** extremal graph which is K_3–free. That is,

$$T_2(n) = \{K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\}.$$

The Theorem of Turán

It is clear that any r–partite graph G
does not contain the complete graph K_{r+1}.

Turán described the structure of he r-partite graphs with order n
and maximum edge density.

The **Turán graph** $T_r(n)$ is an r-partite complete graph
with stable sets $V_1, V_2, \ldots, V_r$,
with cardinalities as close as possible one to each other.

The Theorem of Turán

That is,

$$V(T_r(n)) = V_1 \cup V_2 \cup \ldots \cup V_r,$$

such that

$$\lfloor \frac{n}{r} \rfloor \le n_i \le \lceil \frac{n}{r} \rceil,$$

where $n_i = |V_i|, \quad 1 \le i \le r.$

- If $n_j \ge n_i + 2, \quad 1 \le i < j \le r,$

  by moving one vertex from V_j to V_i we have

  $$|V_j'| \ge n_j - 1, \qquad |V_i'| = n_i + 1.$$

  Therefore,

  $$\begin{aligned} |E(V_i', V_j')| &\ge (n_j - 1)(n_i + 1) = n_i n_j + (n_j - n_i - 1) \\ &> |E(V_i, V_j)|. \end{aligned}$$

  and the graph would not be extremal.

The Theorem of Turán

In particular,

$$n_i = \lfloor \frac{n+i-1}{r} \rfloor,$$

we have a distribution of stable sets which **uniquely** determines $T_r(n)$.

$$\lfloor \frac{n}{r} \rfloor = n_1 \leq n_2 \leq \cdots \leq n_r = \lceil \frac{n}{r} \rceil.$$

- The extremal graphs for the extremal values of $r \in \{1, n\}$ are

$$T_1(n) = N_n$$

$$T_n(n) = K_n.$$

- It is also clear that

$$T_{n-1}(n) = K_n - e.$$

The Theorem of Turán

- For $2 \leq r \leq n-2$,

  the density of $T_r(n)$ can be trivially bounded by

$$\binom{n}{2} - r\binom{\lceil \frac{n}{r} \rceil}{2} \leq t_r(n) \leq \binom{n}{2} - r\binom{\lfloor \frac{n}{r} \rfloor}{2}.$$

  By the Euclidian division we can compute the exact value of $t_r(n)$.

$$n = r\lfloor \frac{n}{r} \rfloor + k, \;\; 0 \leq k < r,$$

  Then, $T_n(n)$ must have:

  k stable sets with $\lceil \frac{n}{r} \rceil$ vertices,
  and
  $r-k$ (the remaining ones) stable sets, with $\lfloor \frac{n}{r} \rfloor$ vertices.

The Theorem of Turán

Therefore,

$$t_r(n) = \binom{n}{2} - k\binom{\lceil \frac{n}{r} \rceil}{2} - (r-k)\binom{\lfloor \frac{n}{r} \rfloor}{2}.$$

If r divides n, then all stable sets have the same size n/r and

$$t_r(n) = \frac{(r-1)n^2}{2r}.$$

- Remark: For $r = 2$ we get the Mantel's bound.

The Theorem of Turán

From the definition we have the following relation between

the **maximum** and **minimum degrees** of a Turán graph.

$$\Delta(T_r(n)) = n - \lfloor \frac{n}{r} \rfloor, \qquad \delta(T_r(n)) = n - \lceil \frac{n}{r} \rceil.$$

So,

$$\Delta(T_r(n)) \leq \delta(T_r(n)) + 1.$$

Moreover,

for any graph G of order n and size $t_r(n)$ we always have

$$\delta(G) \leq \delta(T_r(n)) \leq \Delta(T_r(n)) \leq \Delta(G).$$

Exercise

Find a graph G of order $n \in \{4, 5, 6\}$ and size $t_3(n)$ s.t.

$$\delta(G) \leq \delta(T_3(n)), \qquad \Delta(G) \geq \Delta(T_3(n)).$$

The Theorem of Turán

An interesting consequence is the following simple Lemma.

Lemma

When we delete from $T_r(n)$

1. *a vertex with minimum degree, we obtain*

$$T_r(n-1).$$

2. *an stable set of order h, $\lfloor \frac{n}{r} \rfloor \le h \le \lceil \frac{n}{r} \rceil$, we obtain*

$$T_{r-1}(n-h).$$

□

Exercise

Prove this lemma.

The Theorem of Turán

Now we can prove the famous Turán's Theorem:

Theorem (Turán, 1941)

For any pair of natural numbers $2 \leq r \leq n$,

$$EX(n; K_{r+1}) = \{T_r(n)\}.$$

Proof of Turán's Theorem (I)

Proof: Let G be a graph of order n and size $t_r(n)$ s.t.

$$K_{r+1} \not\to G.$$

We want to show that $G = T_r(n)$.

We use induction on n.

- For $n \leq 3$
  the result clearly holds, since,

$$T_1(n) = N_n \qquad T_n(n) = K_n, \qquad T_{n-1}(n) = K_n - e.$$

Proof of Turán's Theorem (II)

- Let $n \geq 4$, and $2 \leq r \leq n-2$.

  Assume as IH:
  for every n', $r+1 < n' < n$, we have

  $$EX(n'; K_{r+1}) = \{T_r(n')\}.$$

  Let $x \in V(G)$ a vertex of minimum degree.

  Since

  $$\delta(G) \leq \delta(T_r(n)),$$

  $G' = G - x$ has size

  $$m' = t_r(n) - \delta(G) \geq t_r(n) - \delta(T_r(n)) = t_r(n-1).$$

  Therefore,
  since $m' \leq t_r(n-1)$, we have

  $$m' = t_r(n-1).$$

Proof of Turán's Theorem (III)

On the other hand,

$$K_{r+1} \not\to G',$$

since $K_{r+1} \not\to G$.

Therefore, by IH we have,

$$G' = T_r(n-1).$$

Let us now to show that $G = T_r(n)$.

Since

$$K_{r+1} \not\to G$$

a vertex x can not be adjacent to all the stable sets of $T_r(n-1)$.

Therefore,

x belongs to one of the stable sets.

Proof of Turán's Theorem (and IV)

On the other hand,

since x is adjacent to $\delta(G) = \delta(T_r(n)) = n - \lceil \frac{n}{r} \rceil$ vertices,

x is *not* adjacent to

$$(n-1) - (n - \lceil \frac{n}{r} \rceil) = \lceil \frac{n}{r} \rceil - 1 = \lfloor \frac{n}{r} \rfloor,$$

vertices of a small stable set of G'.

Hence,

when we add x to this stable set we obtain $T_r(n)$ as desired. □

Another view of Turan's Theorem

Observe that,

if $n > r$,

the addition of any edge to $T_r(n)$ results in a graph

which contains K_{r+1} as a subgraph.

$$K_{r+1} \hookrightarrow T_r(n) + e$$

Therefore

$$t_r(n) + 1$$

is the **minimum size** of a graph G of order n

which contains K_{r+1} as a subgraph.

$$K_{r+1} \hookrightarrow G$$

The forbidden subgraph problem

Now, we present the general and fundamental theorem for

The forbidden subgraph problem

The only way to force a graph F (with the only exception of a forest) to be a subgraph of a given graph G of order n
is to force its **global density:**
The **global density** of a graph G of order n and size m is

$$0 \leq \frac{m}{\binom{n}{2}} \leq 1.$$

- We say that a graph is **dense** if its size is a **quadratic function on its order**.
- There are graphs with large average degree, connectivity, ... without a prescribed subgraph.

The forbidden subgraph problem

How many edges will suffice to force a graph F to be a subgraph of any graph G on n vertices?
(No matter how these edges are arranged.)

Questions:

- What will such a graph look like?
- Will it be unique?

Remark

The classical theorem of Turán served as a **model** *to go further into this problem when the forbidden subgraph F is not a complete graph.*

The theorem of Erdős-Stone

The **complete multipartite graph** K_r^s
has r stable sets,

$$V(K_r^s) = V_1 \cup V_2 \cup \ldots \cup V_r,$$

s.t.

$$|V_1| = \cdots = |V_r| = s,$$

and for $1 \leq i \leq r$,
each vertex in V_i is adjacent to all the vertices in $V \setminus V_i$.

Theorem (Erdős-Stone, 1946)

For all natural numbers $r \geq 2$ and $s \geq 0$ and every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every graph with $n \geq n_0$ vertices and at least

$$t_r(n) + \epsilon n^2$$

edges contains K_r^s as subgraph.

Comments to the theorem of Erdős-Stone

- It is **surprising** that,
  for any $s > 1$,
  just ϵn^2 more edges (for a fixed $\epsilon > 0$ and n large)
  gives a K_r^s.

- This theorem is **fundamental** because
  it gives a precise asymptotic information for
  any forbidden subgraph F at once.

- Its original proof is **very technical**.
  A simpler more recent proof
  uses a very important and deep result:

The regularity lemma

Szemerédi's Regularity Lemma

Lemma (**Szemerédi, 1971**)

$\forall \epsilon > 0$ and $\forall n \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that every graph G with $|V(G)| \geq n$ admits an **ϵ-regular partition $\mathcal{P}$** *of $V(G)$ with $|\mathcal{P}| = k$ with*

$$n \leq k \leq N.$$

Remark

1. *Every graph has an ϵ-regular partition into a bounded number of sets.*
2. *N ensures that if $|V(G)|$ is large k is also large.*
3. *If each part is large then the result is powerful.*
4. *n is a lower bound for $|\mathcal{P}| = k$.*
5. *If n is large, the number of edges between the parts increase.*
6. *It is useful for dense graphs.*

The meaning of the Regularity Lemma

All connected graphs can be partitioned into a bounded number of equals parts, such that

- most of its edges go between different parts, and
- the edges between any 2 different parts are **fairly uniformly distributed** (as we would expect if they had been generated at random.)

In other words,
all sufficiently large graphs
can be approximated by a random–like graph.

Regular pairs

Let $G = (V, E)$, $X, Y \subset V$, $X \cap Y = \emptyset$

The **edge-density** of the pair (X, Y) is

$$d(X, Y) = \frac{|E(X, Y)|}{|X||Y|}.$$

Definition

Let $\epsilon > 0$. The pair (A, B) of disjoint subsets of vertices is **ϵ-regular** if, for each pair

$$X \subset A, \ |X| \geq \epsilon|A| \ \text{and} \ Y \subset B, \ |Y| \geq \epsilon|B|,$$

the density of (X, Y) differs from the one of (A, B) by at most

$$|d(A, B) - d(X, Y)| \leq \epsilon$$

Remark

The smaller the ϵ more uniform is the distribution.

Regular partitions

Let

$$\mathcal{P} = \{V_0, V_1, \cdots, V_k\}$$

be a partition of V.

Definition

The partition $\mathcal{P}$ is ϵ-**regular** if

1. $|V_0| \leq \epsilon |V|$.
2. $|V_i| = |V_j|, \quad 1 \leq i < j \leq k$.
3. $(V_i, V_j), \quad 1 \leq i < j \leq k$, are ϵ-regular except for at most ϵk^2 pairs.

Remark

V_0 makes (b) possible. It is an **exceptional set** *(can be $\emptyset$)*

Szemerédi's Regularity Lemma

Lemma (**Szemerédi, 1971**)

$\forall \epsilon > 0$ *and* $\forall n \in \mathbb{N}$, $\exists N \in \mathbb{N}$ *such that every graph* G *with* $|V(G)| \geq n$ *admits an* **ϵ-regular partition** $\mathcal{P}$ *of* $V(G)$ *with* $|\mathcal{P}| = k$ *with*

$$n \leq k \leq N.$$

Remark

1. *Every graph has an* ϵ*-regular partition into a bounded number of sets.*
2. N *ensures that if* $|V(G)|$ *is large* k *is also large.*
3. *If each part is large then the result is powerful.*
4. n *is a lower bound for* $|\mathcal{P}| = k$.
5. *If* n *is large, the number of edges between the parts increase.*
6. *It is useful for dense graphs.*

The theorem of Erdős-Stone

The **complete multipartite graph** K_r^s
has r equally large stable sets,

$$V(K_r^s) = V_1 \cup V_2 \cup \ldots \cup V_r,$$

$$|V_1| = \cdots = |V_r| = s,$$

and each vertex in V_i is adjacent to all the vertices in $V \setminus V_i$.

Theorem (Erdős-Stone, 1946)

For all natural numbers $r \geq 2$ and $s \geq 0$ and every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every graph with $n \geq n_0$ vertices and at least

$$t_r(n) + \epsilon n^2$$

edges contains K_r^s as subgraph.

Extremal problems and chromatic number

Next Corol.lary
makes the Erdős-Stone theorem **the most important one**
in Extremal Graph Theory.

Consider
the maximum edge-density
that a graph G without F as a subgraph can have:

$$0 < \frac{ex(n,F)}{\binom{n}{2}} \leq 1$$

Corollary (Fundamental result of extremal graph theory)

For every graph F with at least one edge

$$\lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{2}} = \frac{\chi(F)-2}{\chi(F)-1}$$

Some comments on the fundamental result

Remark

The limit

$$\lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{2}}$$

depends on an invariant of the graph F: its chromatic number, $\chi(F)$.

Remark

We need $ex(n,F)$ to be close to $t_r(n)$ when $r|n$.

Lemma

For $t_r(n)$ we effectively have

$$\lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} = \frac{r-2}{r-1}.$$

Some comments on the fundamental result

Remark

If we can not color F with r colors, then for any $n \in \mathbb{N}$,

$$F \not\to T_r(n).$$

Therefore,

$$t_r(n) \leq ex(n, F).$$

Remark

If $F \subset K_r^s$ for all s large enough, then

$$ex(n, F) \leq ex(n, K_r^s).$$

In particular,

if F is **bipartite** *then*
much less than $\binom{n}{2}$ edges suffice to force F to be in G.

Proof of the fundamental Corollary (I)

Fix a large natural number s.

Let $n \in \mathbb{N}$ (large) and $\epsilon > 0$.
The Erdős-Stone theorem: $ex(n, K_r^s) < t_r(n) + \epsilon n^2$.

Therefore,

$$\begin{aligned}
\frac{t_r(n)}{\binom{n}{2}} &\leq \frac{ex(n,F)}{\binom{n}{2}} \\
&\leq \frac{ex(n,K_r^s)}{\binom{n}{2}} \\
&< \frac{t_r(n)}{\binom{n}{2}} + \frac{\epsilon n^2}{\binom{n}{2}} \\
&= \frac{t_r(n)}{\binom{n}{2}} + \frac{2\epsilon}{1-\frac{1}{n}} \\
&\leq \frac{t_r(n)}{\binom{n}{2}} + 4\epsilon
\end{aligned}$$

Proof of the fundamental Corollary (and II)

Since,

$$\lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} = \frac{r-2}{r-1},$$

we have,

$$\lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{2}} = \frac{r-2}{r-1}.$$

Actual values of $ex(n, F)$

We will show actual values of $ex(n, F)$ when F is a path or a cycle later on.

For trees:

Conjecture (Erdős-Sós, 1963)

For all trees T with $k \geq 2$ edges,

$$ex(n, T) \leq \frac{1}{2}(k-1)n.$$

Remark

This bound is best possible.

Extremal problems on paths and cycles.

We know that:

trees are graphs maximally acyclic, that is,

if we add any edge we create a cycle.

Therefore,

any graph that has size at least its order,

contains at least a cicle.

In general, a graph is

extremal with respect some property (or parameter)

if it has the maximum (or minimum) value for this parameter as

a function of other fixed parameters in the graph.

Extremal problems for the family of cycles.

For instance,

any connected acyclic graph of order n has at most $n-1$ edges.

That means that

trees are the extremal graphs with the property of being acyclic.

On the other,

any graph of order $n \geq 3$ and size at least n, has a cycle.

Therefore,

the **cycles** are extremal graphs with this property.

Extremal problem for C_4

Next result gives the extremal value for forbidden C_4.

Theorem

Let G be a graph with $n \geq 4$ vertices and m edges. If

$$m \geq \frac{n + n\sqrt{4n-3}}{4} + 1$$

then G contains a C_4.

Remark

- *To ensure a triangle we need $(n^2/4) + 1$ edges.*
- *To ensure a C_4 we just need $cn^{3/2}$.*

Proof of extremal problem for C_4 (I)

Proof:

Suppose G does not contain C_4.

- For each vertex x
  the number of pairs of vertices adjacent to x is
  $$\binom{d(x)}{2}$$
  .
- Since G does not contain C_4
  no pair of vertices is counted more than once in the sum below
  $$\sum_{x \in V} \binom{d(x)}{2} \leq \binom{n}{2}.$$

Proof of extremal problem for C_4 (and II)

- As we have seen in the above session,

$$\sum_{x \in V} \binom{d(x)}{2} \geq \frac{m}{n}(2m - n).$$

- Combining the two inequalities,

$$m \leq \frac{n(1 + \sqrt{4n - 3})}{4}.$$

□

Exercise

Find $ex(n, K_{2,3})$.

Pancyclicity of dense Hamiltonian graphs

If G has the density which ensures the existence of a
triangle
and
it is hamiltonian,

then it is pancyclic.

Theorem

Let G be a hamiltonian graph with n vertices and m edges.

If $m > n^2/4$ then G has cycles of all lengths.

Proof of Pancyclicity (I)

Proof:

The proof is by induction on n.
When $n = 3$ there is nothing to prove.

- Suppose first that G has a cycle C' of length $n-1$.
  Thus $G - x$ is hamiltonian for some $x \in V(G)$.
  - If $\delta(x) < n/2$ then

    $$|E(G')| > n^2/4 - (n-1)/2 > (n-1)^2/4$$

    and G' is pancyclic by IH.
  - If $\delta(x) \geq n/2$ then
    x has two neighbours in the $(n-1)$–cycle C' at distance k in the cycle for each $k = 1, 2, \ldots, (n-1)/2$,
    thus providing cycles of all remaining lengths.

Proof of Pancyclicity (and II)

- Suppose G has no cycle of length $n-1$.

  Let $x_1, x_2, \ldots, x_n$ be the hamiltonian cycle.

  - For each i and j at most one of the two edges

    $$x_i x_j \qquad\qquad x_{i+1}x_{j+2}$$

    belongs to G (otherwise G contains an $(n-1)$–cycle).
  - Therefore

    $$d(x_i)+d(x_{i+1}) \le n$$

    which implies $\delta(G) \le n/2$ and

    $$2|E(G)| = \sum_{i=1}^{n} d(x_i) \le n^2/4,$$

    a contradiction. □

Extremal problems for paths and cycles.

For the next extremal results on paths and cycles
we recall the Theorem of Pósa (extension of Dirac's Theorem):

Theorem (Pósa, 1952)

Let G be a graph with $n \geq 3$ vertices.

If for every pair of non adjacent vertices

$$d(x) + d(y) \geq k,$$

then G contains

- *A path of length at least k and*
- *A cycle of length at least $(k+1)/2$.*

Extremal problems for paths and cycles.

The next result gives an optimum **upper bound** for the density of a graph in terms of the maximum lengths of paths.

Theorem

Let G be a graph of order n without paths of length k.
Then,

1. $|E(G)| \leq n(k-1)/2$.
2. *G is extremal if and only if all its components are K_k.*

Proof:

If $k = 1$ then $G = N_n$.

We fix $k \geq 2$.
If $n \leq k$, then

$$|E(G)| \leq (n-1)n/2$$

and G is extremal if and only if $G = K_n = K_k$.

Paths and cycles: Proof of Theorem (I)

For $2 \leq k < n$ we use induction on n.

- The result is trivial for $n \leq 3$.
- Let $n > 3$.
  - Suppose first that G is not connected.

    Let $C_1, C_2, \cdots, C_s$ be the connected components of G,

$$\begin{aligned} |E(G)| &= \sum_{i=1}^{s} |E(C_i)| \\ &\leq \sum_{i=1}^{s} \frac{|V(C_i)|}{2}(k-1) \qquad (IH) \\ &= n(k-1)/2. \end{aligned}$$

    So, the extremal graphs are $G = \cup K_k$ and size

$$|E(sK_k)| = sk(k-1)/2 = n(k-1)/2.$$

Paths and cycles: Proof of Theorem (and II)

- Let $n > 3$.
  - Suppose now G is connected.

    Pósa's Theorem implies the existence of a vertex x in G s.t.

    $$d(x) \leq (k-1)/2,$$

    otherwise there would be a path of length k.
    Using induction on $G - x$ we have,

    $$\begin{aligned} |E(G)| &= d(x) + |E(G-x)| \\ &\leq (k-1)/2 + (k-1)(n-1)/2 \\ &= n(k-1)/2. \end{aligned}$$

    If there is equality then $G - x$ is an extremal graph,
    and by IH it must be a union of K_k's.
    But then G has a path of length k (contradiction). □

Extremal problem for the trees

The next result gives a global **lower bound** for the density of a graph which contains any tree of a given size.

Theorem

Let $1 \leq k < n$. If G has order n and size

$$m \geq (k-1)n - \binom{k}{2} + 1.$$

then G contains all trees of size k.

Remark

- *Recall that the Erdős-Sós conjecture says that the extremal value for a tree with k edges is $(k-1)n/2$.*
- *For stars and paths we have seen that the Erdős-Sós conjecture holds.*

Proof of bound for extremal problem for the trees (I)

Proof:

We first prove that
the bound on m guarantees that $\exists\ H \hookrightarrow G$ s.t $\delta(H) \geq k$.

- By induction on n.
- If $n = k + 1$

$$m \geq (k+1)(k-1) - \binom{k}{2} + 1 = \binom{k+1}{2},$$

Then, $G = K_{k+1}$ and we can take $H = G$.

- Suppose $\delta(G) < k$ (otherwise $H = G$)
let $G' = G - x$ with $d(x) < k$,

$$|E(G')| \geq |E(G)| - (k-1) \geq (k-1)(n-1) - \binom{k}{2} + 1,$$

and by IH: $H \hookrightarrow G'$ s.t $\delta(H) \geq k$.

Proof of bound for extremal problem for the trees (II)

Now,
we want to see that a graph H with $\delta(H) \geq k$
contains every tree of size k: $T_k \hookrightarrow H$

- By induction on k.
- For $k = 1$ there is nothing to prove.
- Let T be a tree with k edges and x a leave of T.
- By IH, H contains $T' = T - x$.
- If y is the vertex incident to x in a copy of $T' \hookrightarrow H$
  it has degree at least k in H.
  Since T' has $k - 1$ edges,
  then $T = T' + xy \hookrightarrow H \subset G$. □

Extremal problem for the circumference

Next result,
relates the maximum size of a graph
in terms of the length of its circumference.

Theorem

Let G be a graph of order n such that any of its cycles has length at most $3 \leq k \leq n$. Then

$$m \leq k(n-1)/2.$$

□

Exercise

G is extremal if and only if it is connected and all its blocs are K_k.

Extremal problems for Hamiltonian cycles.

We also know that

the existence of a **hamiltonian cycle** is a much difficult problem.

In general, we need a very **high density** in the graph.

Next theorem says precisely

which must be this density as a function of the order of the graph.

Theorem

If G is a graph with $n \geq 3$ and

$$m \geq \binom{n-1}{2} + 2,$$

Then, G is hamiltonian.

Exercise

Find the extremal graphs. □

Extremal problems for Hamiltonian paths.

If we reduce in one unit this size then,

we can not say if the graph is hamiltonian,

but at least we know that it has a hamiltonian path.

Theorem

If G is a graph with order $n \geq 2$ and size

$$m \geq \binom{n-1}{2} + 1,$$

then, G has a hamiltonian path.

Exercise

Find the extremal graphs.

Extremal Problem for the Diameter and for the Girth

Now we consider another extremal problem
(that you already know).

We want to maximize/minimize the **order** of a graph G with respect $\Delta(G)$ and $D(G)$.

The extremal graphs are called Moore graphs,

$$M(\Delta, D),$$

have

1. maximum degree Δ
2. diameter D, and
3. the maximum possible order,

$$N(\Delta, D) = 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1}.$$

Extremal Problem for the Diameter and order

- $M(\Delta, 1) = K_{\Delta+1}$.
- $M(2, D) = C_{2D+1}$.
- $M(3, 2) =$Petersen graph.
- For $D = 2$ and $\Delta(G) \geq 3$,
  $M(7, 2)$ and **maybe** (the monster), $M(57, 2)$.

Lemma

The Moore's graphs are regulars. □

Extremal Problem for the Girth and order

Now, we are going to the opposite direction

Problem

What is the **minimum possible order** *that a graph with minimum degree $\delta(G) = r$ and girth $g \geq 3$ can have?*

This minimum order is denoted by

$$n_0(r, g)$$

and the graphs which attain this order are called **cages**.

$$c(r, g).$$

Cages

- For $\delta = 2$,

$$c(2, g) = C_g.$$

- For $g = 3$,

$$c(r, 3) = K_{r+1}.$$

In general, to determine the order of these graphs is a very difficult OPEN problem.
Next theorem give us a lower bound for $n_0(\delta, g)$.

Theorem

Any graph with minimum degree $\delta \geq 3$ and girth $g \geq 3$ has order at least

$$n_0(\delta, g) \geq 1 + \delta + \delta(\delta - 1) + \cdots + \delta(\delta - 1)^c$$

where

$$c = \lfloor (g-1)/2 \rfloor.$$

Proof of lower bound for cages (I)

Let G be a graph with

$$\delta(G) \geq 3, \qquad g(G) \geq 3.$$

Fix a vertex $x \in V(G)$.

We prove by induction that the number of vertices at distance k of x, $1 \leq k \leq c = \lfloor (g-1)/2 \rfloor$, is at least

$$\delta(\delta - 1)^{k-1}.$$

For $k = 1$, the result is obvious.

Proof of lower bound for cages (and II)

For $k > 1$,

Let $y \in V(G)$ such that $1 < d(x, y) = k < c$.

1. there is an only vertex adjacent to y which is at distance $k - 1$ of x, (otherwise we find a cycle of length $2k < g$), and
2. there is no vertex at distance k,
   (otherwise we find a cycle of length $2k + 1 < g$).

Therefore,

1. y has at least $\delta - 1$ neighbors at distance $k + 1$ of x, and
2. any vertex z adjacent to y at distance $k + 1$ of x, admits only one adjacent vertex, y, at distance k of x.
   (since $2(k + 1) \leq 2c < g$).

Hence the number of vertices at distance $k + 1$ from x is at least

$$\delta(\delta - 1)^{k-1}(\delta - 1).$$

Regularity and Cages

Using the previous proof and the fact that the vertices at maximum distance of x must be adjacent between them and well distributed, try to prove next Corollary.

Corollary

Let G be a graph with minimum degree $\delta \geq 3$ and girth $g \geq 3$. If G has order $n_0(\delta, g)$, then G is δ-regular. □

Extremal graphs for the diameter and girth

Recall that the minimum length of a cycle is upper bounded by

$$g \leq 2D + 1.$$

In particular, it is easy to prove that if a graph is extremal with respect to the minimum order, then the girth is maximum as a function of the diameter, that is,

$$g = 2D + 1.$$

Exercise

Let G be an r-regular graph with girth g and minimum order.

1. *If $g(G) = 2k + 1 \geq 3$ then G has diameter k.*
2. *If $g(G) = 2k \geq 4$ then G has diameter $k + 1$.*

□

Extremal graphs for the diameter and girth

As a consequence, if G is extremal with $g = 2k + 1 \geq 3$, then G is r-regular, has diameter k and therefore,

G is the Moore graph $M(r, k)$.

We have reviewed Moore graphs (extremal graphs with odd girth).

For the extremal graphs of even girth $g = 2k$, we have some results.

- for $r \geq 4$,
$$c(r, 4) = K_{r,r}.$$
- If $g = 6$ then, there is one cage r-regular if there exists a projective plane of order $r - 1$.
- For $g = 8$ and $g = 12$ its existence is related to the existence of a certain projective geometry.

Exercise

If G is an extremal graph with $g = 2k \geq 4$ then each vertex of G is at distance $k - 1$ of each pair of adjacent vertices. □

Extremal problems for connectivity

Recall

- A graph is k–connected if the removal of any set of k vertices does not disconnect the graph.
- Equivalently, any two vertices are connected by at least k internally disjoint paths (Theorem of Menger).
- The connectivity $\kappa(G)$ is the minimum k such that there is a set of k vertices which disconnects de graph.
- Equivalently $\kappa(G) = k$ means that G is $(k-1)$–connected but not k–connected.

Extremal problems for connectivity

The **connectivity**

$$\kappa(G),$$

of a graph G, is one of the most important parameters in a graph, (which usually is very difficult to determine).

A graph of order n and connectivity k is **extremal** (for the connectivity) if it has the **minimum number of edges**, which is denoted by

$$m_0(n, k).$$

Extremal graphs with connectivity one

If $\kappa(G) = 1$ then, G has at least 1 cut vertex.

Therefore, their blocks (maximal 2–connected subgraphs) are structured like a tree.

The only extremal graphs with

$$\kappa(G) = 1$$

and order n are the **trees** and so

$$m_0(n, 1) = n - 1.$$

Extremal graphs with connectivity two

If $\kappa(G) = 2$ and G is extremal

1. none of its vertices can disconnect the graph, and
2. every pair of vertices lay on a cycle.

Therefore, its set of blocks is structured in a cyclic way, and the only extremal graphs are the **cycles**.

Therefore

$$m_0(n, 2) = n.$$

Extremal graphs with connectivity three

The graphs with $\kappa(G) = 3$ may have a complex structure.

- The smallest 3–connected graph is K_4
  It can be obtained by adding a vertex x to C_3 and joining it to all vertices of the triangle:
  $$K_4 = C_3 + x.$$
- For $n \geq 3$,
  the wheels $W_n = C_n + x$ are graphs with low edge density and
  $$\kappa(W_n) = 3$$
  But, these 3-connected graphs with $|E(W_n)| = 2n$ are not always extremal. For example,
  - $K_3 \times K_2$ is 3-connected, have order 6 and size 9.
  - For $n \geq 6$, we have $\kappa(K_{3,n-3}) = 3$ and $|E(K_{3,n-3})| = 3(n-3)$, which is less than $2n$ if $n < 9$.

Minimally k–connected graphs

To determine the minimum absolute size,

$$m_0(n, k),$$

that a graph of order n and connectivity $k > 2$ can have is a very difficult OPEN problem.

A way to approximate the problem consists in studying the graphs with connectivity k and order n which have the **minimal size**.

Definition

A graph G is **minimally k-connected** if

1. $\kappa(G) = k$, and
2. for any edge $e \in E(G)$,

$$\kappa(G - e) = k - 1$$

Minimally k–connected graphs

- It is clear that any k-connected graph admits a minimally k-connected subgraph.
- The connectivity of a graph G is the minimum of the connectivities between each pair of vertices, that is,

$$\kappa(G) = \min\{\kappa(x, y) : \ x, y \in V(G)\},$$

where $\kappa(x, y)$ is the minimum cardinality of a set which separates x from y.

Lemma

A k-connected graph is minimally k-connected if and only if for each pair x, y of adjacent vertices, $\kappa(x, y) = k$.

Proof of Lemma

Proof:
The condition is sufficient (clear from the definition).

On the other hand,
if G is a graph minimally k-connected,
then for any edge $xy \in E(G)$,

$$\kappa(G - xy) = k - 1$$

Hence there is a cutset with cardinality $k - 1$ in $G - xy$
which separates x from y.

Therefore, there are at most $k - 1$ independent xy-paths and, since $\kappa(G) = k$, we get

$$\kappa(x, y) = k.$$

Halin's theorem

We know that in any graph G we always have

$$\kappa(G) \leq \delta(G).$$

Exercise

Give an example of a cubic graph G with connectivity 1:
$\delta(G) = 3 > \kappa(G) = 1$.

Halin's Theorem says that we have equality if G is minimally k–connected.
To prove it we will use the following result (Exercise)

Lemma

A graph G is k-connected if and only if

1. *it has at least $k+1$ vertices and*
2. *for each $V' \subset V(G)$ with $|V'| = k$ and each $x \in V(G) \setminus V'$, there exist k paths from x to V' which only intersect in the vertex x.*

□

Halin's theorem and its proof (I)

Theorem (Halin, 1969)

If G is a graph minimally k-connected, then

$$\kappa(G) = \delta(G).$$

Proof:
Let $G \neq K_n$ be a minimally k-connected graph of order $n \geq k+2$.

We will show that

Claim

There is a k–cutset T such that $G - T$ has a connected component with an only vertex.

Halin's theorem follows from the claim.

Proof of Halin's theorem (II)

Proof

- Choose a cutset T with cardinality k which minimizes the cardinality of the smallest connected component of $G - T$.
- Let
$$\mathcal{C} = \{V_1, V_2, \cdots, V_{s\geq 2}\}$$
be the set of components of $G - T$ ordered by increasing cardinality,
$$|V_1| \leq |V_2| \leq \cdots \leq |V_s|.$$
- Suppose on the contrary that $|V_1| \geq 2$.
Thus there is $xy \in E(G)$ in V_1.
Since G is minimally k-connected,
$\exists$ a cutset $T' \subset G - xy$ s.t.
$$|T'| = k - 1$$
which separates x from y.

Proof of Halin's theorem (III)

Claim B

$$V' = V_2 \cup \cdots \cup V_r \subset T'.$$

Suppose on the contrary there is $z \in V' \setminus T'$.

By the above Lemma we know that, for each vertex $u \in G - T$ there is a family F_u of k paths from u to T which only intersect in u.

- Note that, the paths of F_x do not contain the edge xy
  Otherwise we can separate x from V' with a cutset T' with cardinality $(k-1)$ and $T' \cup \{y\}$ would be a cutset with cardinality k leaving a smaller connected component $V_1' = V_1 - y$.
- Similarly, the paths of F_y do not contain the edge xy.
- The paths of F_z clearly do not contain the edge xy.

Proof of Halin's theorem (IV)

Claim B

$$V' = V_2 \cup \cdots \cup V_r \subset T'.$$

Hence $\exists$ k internally disjoint paths from x to z and also from y to z in $G - xy$:
it is not possible to separate x from y with a cutset T' with cardinality $(k-1)$, a contradiction.

Proof of Halin's theorem (and V)

Let $D = T \setminus T'$ and set $t = |D|$.

For each $v \in D$ at least one of the 2 paths,

from x to v in F_x or from y to v in F_y

contains some vertex of T'
(because this set separates x from y).

Therefore,

$$|T' \cap V_1| \geq t/2$$

and as $V' \subset T' \setminus T$ we have,

$$|V'| \leq k - 1 - (k - t) - t/2 = t/2 - 1 < |T' \cap V_1| \leq |V_1|,$$

which contradicts the minimality of V_1. □

A theorem of Mader

Next theorem, proved by W. Mader, is the most powerful result to get a characterisation of the structure of the minimally connected graphs.

Theorem (Mader, 1972)

If G is a graph minimally k-connected and

$$K = \{x \in V(G) : d(x) = k\},$$

then,

the graph $G - K$ is a forest. □

Here we do not include its proof,
which is long and essentially technical.

A consequence of Mader's theorem

Corollary

If G is a graph minimally k-connected then, for any subgraph H of G we have,

$$\delta(H) \leq k.$$

Proof:

If $\delta(H) > k$ the Theorem of Mader implies that H is a forest. But then $\delta(H) = 1$, a contradicition. □

An upper bound on the size

We finish this section with another result of Mader which gives an **upper bound** for the size of a minimally connected graph whenever its order is large as a function of the connectivity.

The proof is very technical and relies on the above Theorem of Mader.

Theorem (Mader)

If G is a minimally k-connected graph of order $n \geq 3k-2$ then,

$$|E(G)| \leq k(n-k).$$

□

Some comments on the last Mader's Theorem

Remark

This bound is tight:

- *It is easy to check that the graphs*

$$K_{k,2k-2}, \qquad C_{2k}+N_{k-2}$$

  are minimally k–connected, have order $3k-2$ *and size*

$$2k(k-1)=k(n-k).$$

- *On the other hand, if* $n=3k-3$*,*

$$|E(C_{2k-1}+N_{k-2})|=(2k-1)(k-1)=k(n-k)+1.$$

The extremal graph for Mader's Theorem

It is clear that

$$|E(K_{k,n-k})| = k(n-k)$$

If $n \geq 2k$, $K_{k,n-k}$ is minimally k–connected.

Corollary (Mader)

If G is a graph minimally k-connected of order $n \geq 3k-1$ then $|E(G)| = k(n-k)$ iff

$$G = K_{k,n-k}.$$

□

This is the only extremal graph for the previous theorem.