Session 1: Graceful Labelings

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Extremal Problems and Hamiltonicity in Graphs
ITB Bandung, 2-13 February 2009

Overview

Graceful Labelings

Notation: Graph $G = (V, E)$ with n vertices and m edges.

Definition

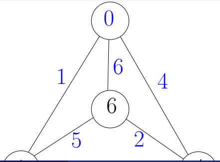
A graceful labeling of G is an injective map

$$f : V \to \{0, 1, \ldots, m\}$$

such that the induced edge values

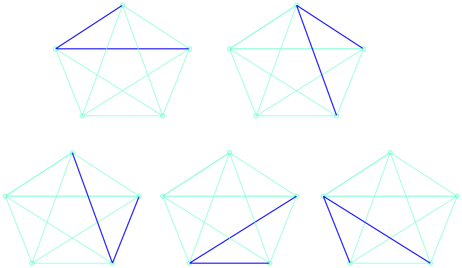
$$\begin{aligned} f_e : E &\to \mathbb{N} \\ xy &\mapsto |f(x) - f(y)| \end{aligned}$$

are pairwise distinct.



Definition

An edge decomposition of a graph H is a partition of its edge set $E(H)$. We say that G decomposes H, and write $G|H$, if H admits an edge decompostion into copies of G.



$P_2|K_5$.

Graceful Labelings and edge decompositions

Proposition

A graceful graph G with m edges decomposes the complete graph K_{2m+1}.

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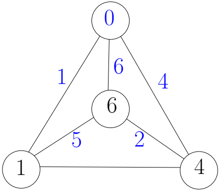

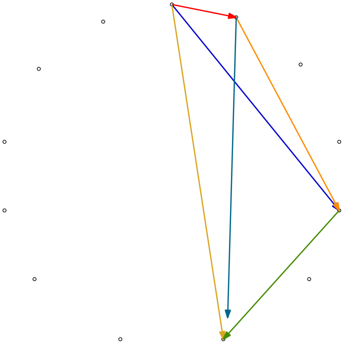

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A graceful graph G with m edges decomposes the complete graph K_{2m+1}.

Lemma

If there is a map $f : V \to [0, k]$ such that the induced edge values $\{|f(x) - f(y)| : xy \in E(G)\}$ are pairwise distinct then there is a $2m$–regular graph H such that G decomposes H.

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Theorem (Wilson, 1975)

For every graph G with no isolated vertices there is k such that G decomposes K_{2k+1}.

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The minimum k for which there is a map $f : V \to [0, k]$ with pairwise distinct edge values is the gracesize $gr(G)$ of G.

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- The wheel W_n is graceful for all $n \geq 4$.Frucht, 1977

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- Tree with at most four ends. Huang, Kotzig, Rosa, 1982
- Tree with at most 27 vertices. Aldred, McKay, 2007

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- Every tree with m edges is a subtree of a graceful tree with at most $3m/2$ edges.
  López, Lladó, 2007

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