Session 2: Modular Labelings

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Overview

Modular Labelings

Definition

A ρ–labeling of G is an injective map

$$f : V \to \mathbb{Z}_{2m+1}$$

such that the set

$$L := \{f(x) - f(y) : \; xy \in E(G)\}$$

of induced edge values verifies
(i) $L \cap -L = \emptyset$, and
(ii) $|L| = m$ (edge labels are pairwise distinct)

Theorem (Rosa, 1967)

A graph G with m edges decomposes cyclically K_{2m+1} of and only if G admits a ρ–labeling.

Definition

A modular bigraceful labeling of a bipartite graph $G = (V_1 \cup V_2)$ is a map

$$f : V \to \mathbb{Z}_m$$

such that
(i) f is injective on each stable set, and
(ii) the set of induced edge labels $L := \{f(x) - f(y) : \ xy \in E(G)\}$ has cardinality m (they are pairwise distinct).

Theorem (López, Lladó, 2007)

A graph G with m edges decomposes cyclically the complete bipartite graph $K_{m,m}$ if and only if G admits a modular bigraceful labeling.

Modular Labelings

Definition

An harmonious labeling of a graph G is an injective (one label can be repeated if G is a tree) map

$$f : V \to \mathbb{Z}_m$$

such that the set of induced edge labels $L := \{f(x) + f(y) : \; xy \in E(G)\}$ has cardinality m (they are pairwise distinct).

Modular Labelings

Definition

A ρ–labeling of G is an injective map $f : V \to \mathbb{Z}_{2m+1}$ such that (i) $L \cap -L = \emptyset$, and (ii) edge labels $L = \{f(x) - f(y) : \ xy \in E\}$ are pairwise distinct.

Definition

A modular bigraceful labeling of a bipartite graph $G = (V_1 \cup V_2)$ is a map $f : V \to \mathbb{Z}_m$ such that (i) f is injective on each stable set, and (ii) edge labels $L = \{f(x) - f(y) : \ xy \in E\}$ are pairwise distinct.

Definition

An harmonious labeling of a graph G is an injective (one label can be repeated if G is a tree) map $f : V \to \mathbb{Z}_m$ such that edge labels $L = \{f(x) + f(y) : \ xy \in E\}$ are pairwise distinct.

Modular Labelings

Conjecture (The Modular Labeling Tree Conjecture)

- *All trees admit ρ–labelings (Ringel-Kotzig, 1966).*
- *All trees are modular bigraceful (López-Lladó, 1995).*
- *All trees are harmonious (Graham-Sloane, 1980).*

Constructing bigraceful trees

Definition

A bigraceful labeling f of a tree T with stable sets A and B is consecutive if $f(A)$ is an interval.

A tree T admits a consecutive bigraceful labeling if and only if T admits an α–labeling.

Constructing bigraceful trees

Definition

A bigraceful labeling f of a tree T with stable sets A and B is consecutive if $f(A)$ is an interval.

Lemma

Let T, T' be bigraceful trees and T consecutive. Then $T[uv]T'$ and $T\{uv\}T'$ are bigraceful, where u, v are vertices where a bigraceful labeling takes value zero.

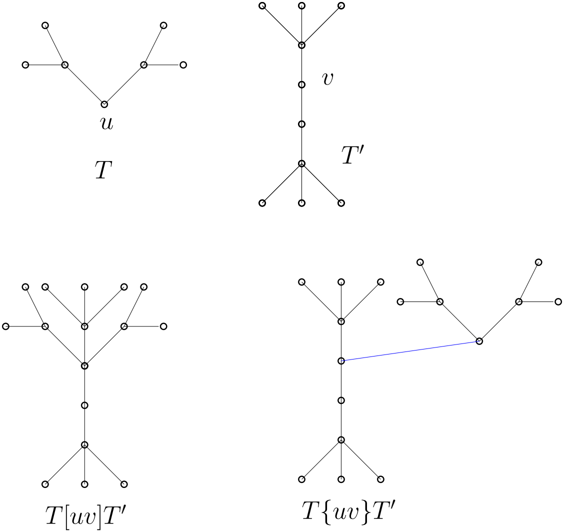
Lemma

Let T be a bigraceful tree and u a vertex where a bigarceful labeling takes the value zero. Then $S_k[u, T]$ is a bigraceful tree.

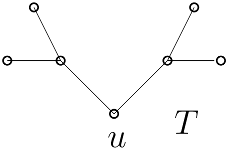
Lemma

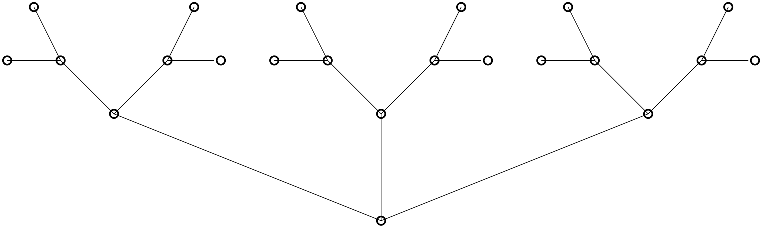
If the base tree of T admits a consecutive bigraceful labeling then T is bigraceful.

Constructing bigraceful trees



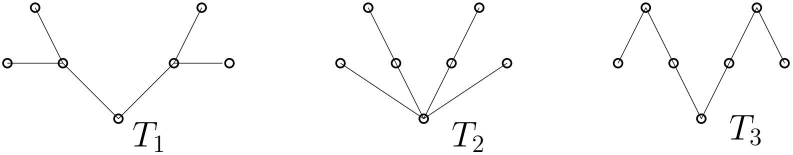
Constructing bigraceful trees

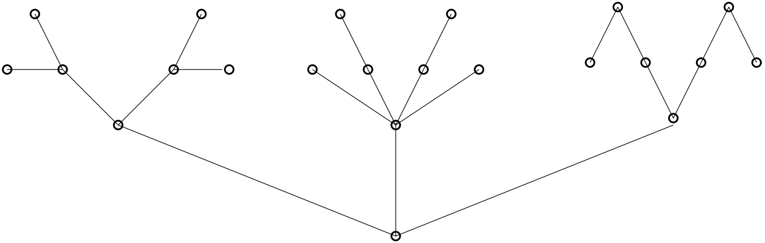




$S_3[uT]$

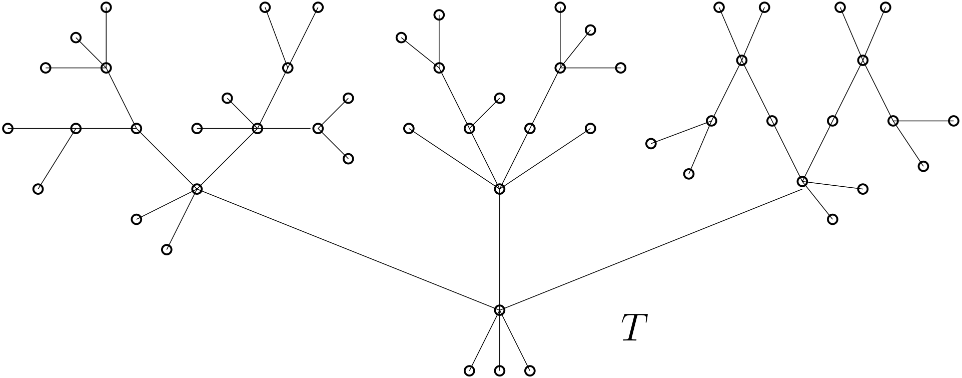
Constructing bigraceful trees



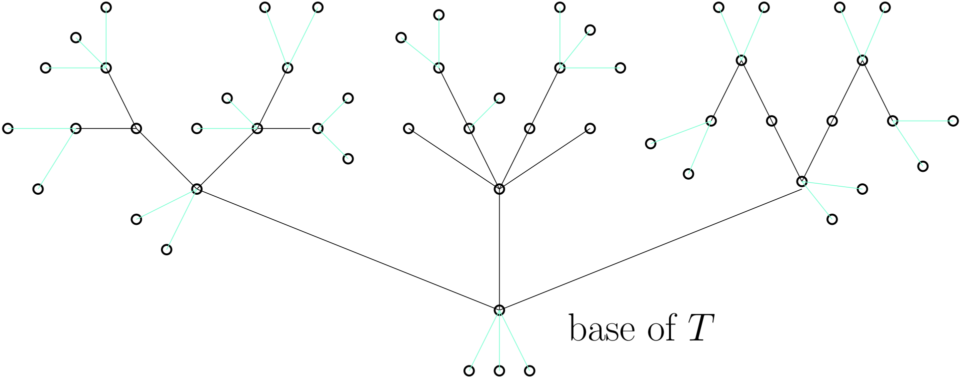


$S_3[u_1T_1, u_2T_2, u_3T_3]$

Constructing bigraceful trees



Constructing bigraceful trees



General results for bigraceful trees

Theorem (López-Lladó, 1995)

The following classes of trees are bigraceful:

- *Lobsters.*
- *Trees of diameter at most five.*
- *Symmetric trees with even degrees.*

Theorem (López-Lladó, 1995)

Any tree is the base tree of a bigraceful tree.
Any tree is homeomorphic to a bigraceful tree.