Session 5: The polynomial method in Graph Labeling

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Overview

The polynomial method

A method based on the use of polynomials which allows one to prove the existence of some combinatorial configurations.

Remark

Let $P(x)$ be a polynomial of degree n with coefficients in a field $\mathbb{F}$. If S is a set of $n+1$ points in $\mathbb{F}$, there is $a \in S$ such that $P(a) \neq 0$.

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Theorem (Alon, 2001)

Let $P(x_1, \ldots, x_k)$ be a polynomial with m variables in a field $\mathbb{F}$. Suppose that P has a monomial $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ of maximum degree. Let $S_1, S_2, \cdots, S_k$ be sets in $\mathbb{F}$. If

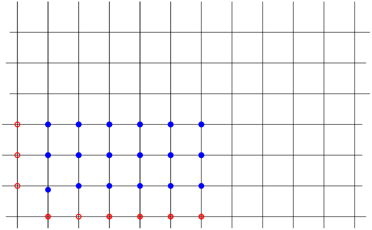
$$|S_1| \geq n_1 + 1, |S_2| \geq n_2 + 1, \ldots, |S_k| \geq n_k + 1,$$

then there is $(a_1, a_2, \ldots, a_k) \in S_1 \times S_2 \times \cdots \times S_k$ such that

$$P(a_1, a_2, \ldots, a_k) \neq 0.$$

An example

$$P(x, y) = x^5y^2 - x^4y^3 + 10x^3y^3 - x^4y^3$$



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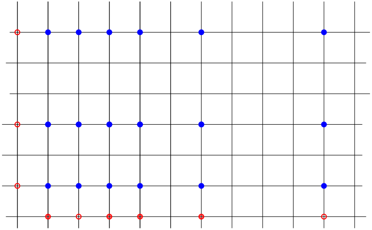

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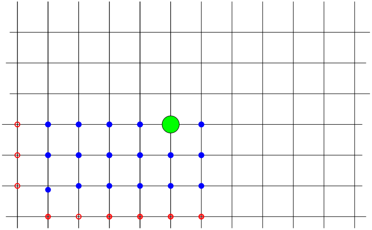

$P(x,y)$ does not vanish only at $(5,3)$ from $S_1 \times S_2$!.

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- Vertex labels x_1, x_2, x_3, x_4

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- Vertex labels x_1, x_2, x_3, x_4
- Vertex labels must be pairwise distinct:

$$P_1(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

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If $P(a, b, c, d) \neq 0$ then a, b, c, d are pairwise distinct.

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- Edge labels:
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$$\begin{aligned} P_2(x_1,x_2,x_3,x_4) &= ((x_1-x_2)^2-(x_2-x_3)^2)((x_1-x_2)^2-(x_3-x_4)^2) \\ &\quad ((x_2-x_3)^2-(x_3-x_4)^2) \end{aligned}$$

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Consider the Polynomial

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&= x_2x_3^5x_4^2 - x_1^4x_4^3x_2 + 4x_1^3x_3^3x_4^2 - 4x_1^3x_3x_4^2x_2^2 + \cdots
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- Consider the monomial $-4x_1^3x_2^2x_3x_4^2$.
- Let $S_1 = \{0,1,2,3,4\}, S_2 = S_4 = \{1,2,3\}$ and $S_3 = \{1,2\}$.

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- Let $S_1=\{0,1,2,3,4\}, S_2=S_4=\{1,2,3\}$ and $S_3=\{1,2\}$.
- By the Combinatorial Nullstellensatz there is a choice of $a\in S_1, b\in S_2, c\in S_3, d\in S_4$ such that $P(a,b,c,d)\neq 0$.

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- By the Combinatorial Nullstellensatz there is a choice of $a \in S_1, b \in S_2, c \in S_3, d \in S_4$ such that $P(a, b, c, d) \neq 0$.

The labeling $x_1 = a, x_2 = b, x_3 = c, x_4 = d$ is a ρ–labeling of P_3.

An application to ρ-labelings

Definition

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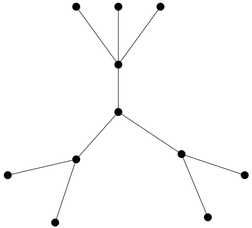

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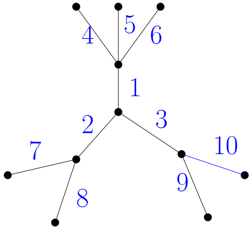

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Theorem (Kezdy, 2006)

Let T be a stunted tree with n edges. If $2n+1$ is a prime then T admits a ρ–labeling.

An application to ρ-labelings

Sketch of the proof

- Order the m edges of T in a ‘stunted’ way.

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Sketch of the proof

- Order the m edges of T in a 'stunted' way.
- Let v_0 be one of the two ends of e_1 and label the vertices: v_i is the vertex in e_i not in $T[e_1, \ldots, e_{i-1}]$.

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- Consider the polynomial

$$P(x_1, \dots, x_n) = \prod_{1 \le i < j \le m} (x_i^2 - x_j^2) \prod_{1 \le i < j \le m} (g_i - g_j) \prod_{i=1}^{m} x_i^i,$$

where $g_i = \sum_{s \in T(i)} x_s$ and $T(i)$ is the set of edges in the only path from v_0 to v_i.

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- Apply Combinatorial Nullstellensatz.

An application to vertex antimagic edge–labelings

Definition

A bijection $f : E \to \{1, \dots, m\}$ is a vertex antimagic edge labeling of a graph $G = (V, E)$ if the vertex sums

$$L = \{\sum_{e \sim v} f(e);\ e \in E\}$$

are pairwise distinct.

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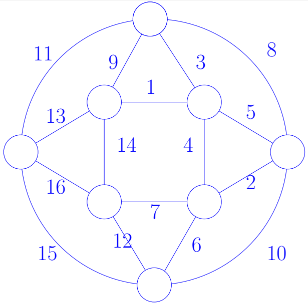

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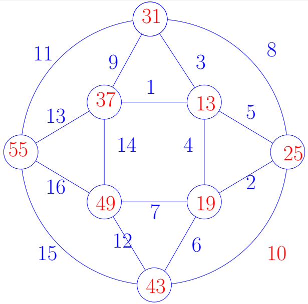

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Conjecture (Ringel (1990))

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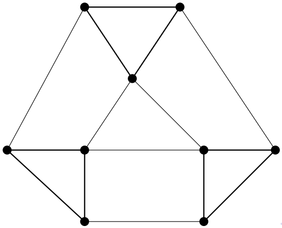
Theorem (Hefetz, 2005)

Let G be a graph with $n = 3^k$ vertices. If G admits a K_3–factor then G admits a vertex antimagic edge labeling.

An application to vertex antimagic edge–labelings

Sketch of the proof



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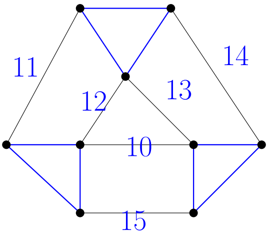
Sketch of the proof

- We pick a K_3–factor F of G and label the edges of $G - F$ arbitrarily with the numbers $\{n+1, n+2, \ldots, |E|\}$.



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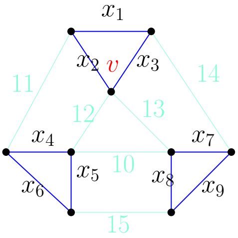
$$P(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j) \prod_{1 \le i < j \le n} (\omega(v_i) - \omega(v_j)),$$

where $\omega(v) = \sum_{e \sim v} f(e)$ (involves two variables).



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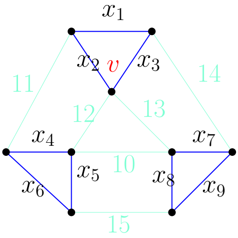
- Check that $x_1^{n-1} x_2^{n-1} \cdots x_n^{n-1}$ has a nonzero coefficient.



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An application to the Graham–Haggkvist conjecture

Recall

Every tree with m edges decomposes the complete bipartite graph $K_{m,m}$

An application to the Graham–Haggkvist conjecture

Let's try

Every tree with m edges decomposes the complete bipartite graph $K_{2m,2m}$

An application to the Graham–Haggkvist conjecture

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Every tree with m edges decomposes the complete bipartite graph $K_{2m,2m}$

Definition

Let $G = Cay(A, S)$ be a bipartite Cayley graph on an abelian group A. A map $f : V \to A$ is a G–bigraceful labeling of a bipartite graph $H = (V, E)$ if (i) it is an injective graph homomorphism and (ii) the edge labels are pairwise distinct.

An application to the Graham–Haggkvist conjecture

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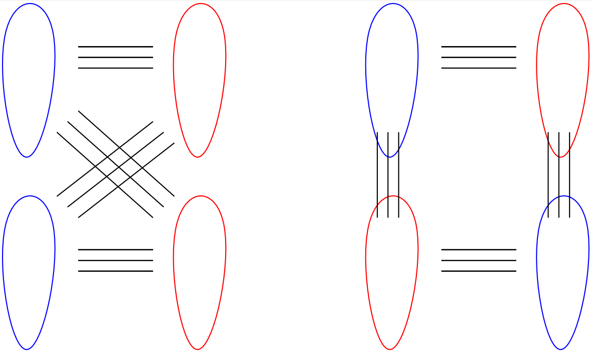
Lemma

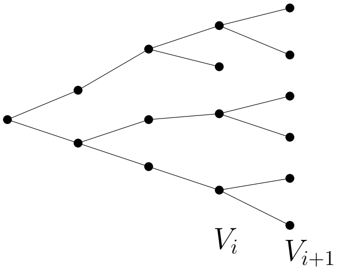
If there is a G–bigraceful labeling of a tree with m edges with $G = Cay(\mathbb{Z}_m \times \mathbb{Z}_4, \mathbb{Z}_m \times \{1\})$ then T decomposes $K_{2m,2m}$.



An application to G–bigraceful labelings

Theorem (Cámara, Lladó, Moragas, 2008)

Let T be a tree with a prime number p of edges. If the growth ratio of T is $\rho(T) \geq \phi^{1/2}$ (ϕ the golden ratio) then T decomposes $K_{2p,2p}$.



$$\rho(T) = \max_{v \in V} \min_i \frac{|V_{i+1}|}{|V_i|}$$

An application to G–bigraceful labelings

Sketch of proof

- Choose a vertex with the desired growth ratio and delete the two last levels of T rooted at v to obtain T_0.

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- Choose a vertex with the desired growth ratio and delete the two last levels of T rooted at v to obtain T_0.
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Sketch of proof

- Choose a vertex with the desired growth ratio and delete the two last levels of T rooted at v to obtain T_0.
- Use the polynomial method to show that T_0 admits a $\mathbb{Z}_p$–bigraceful labeling.
- Use the polynomial method to complete the embedding of T in $\mathbb{Z}_p \times \mathbb{Z}_4$ with edge labels pairwise distinct.

Epilog

- Graph labelings are assignments of elements of a group to elements of a graph verifying some arithmetical properties.

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- The open conjectures on trees reveal the high complexity of this simple family of graphs.
- Labeling graphs is an atractive research topic with 'a wealth of open problems'.